1. Consider the parametric curve defined by $\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(2t) \end{cases}$ for $0 \leq t \leq 2\pi$.

   (a) Find $x'(t)$ and $y'(t)$.

   (b) Find $\dfrac{dy}{dx}$.

   (c) Find the tangent line to the curve at $t = \frac{\pi}{3}$.

   (d) For what values of t is $y'(t) = 0$? For each of these values, find the point in $x-y$ coordinates.

   (e) For what values of t is $x'(t) = 0$? For each of these values, find the point in $x-y$ coordinates.

   (f) Use technology to graph the curve. For each of the points you found above, describe the tangent line.

   (g) Find the speed of the point $\big(x(t), y(t)\big)$ as it moves along the curve at $t = 0$.

2. Consider the parametric curve defined by $\langle e^t - t, t^2 - t\rangle$ for $-1 \leq t \leq 1$.

   (a) For what points on the curve is the tangent line horizontal?

   (b) For what points on the curve is the tangent line vertical?

3. Consider the parametric curve defined by $\begin{cases} x(t) = \sin(t) \\ y(t) = \left(\dfrac{t}{\pi}\right)^2 \end{cases}$ for $-\frac{3}{2}\pi \leq t \leq \frac{3}{2}\pi$.

   (a) For what values of t does the curve intersect itself? (Hint: If $\big(x(t), y(t)\big) = \big(x(s), y(s)\big)$, then $x(t) = x(s)$ and $y(t) = y(s)$. We know $x(t) = x(s)$ if $t = s + 2\pi$, so try solving $y(s + 2\pi) = y(s)$ for a value of s.)

   (b) For one of the values of t above, find the tangent line to the curve. Then find the tangent line for the other value of t.

   (c) Use technology to help you sketch the curve. Draw both tangent lines on the graph.

4. For each part, draw a curve that intersects itself in such a way that:

   (a) the two tangent lines at the intersection are perpendicular.

   (b) for both values of t at the intersection, the tangent line is the same.

   (c) the two tangent lines at the intersection are neither the same, nor perpendicular.

5. *Challenge:* The integrating speed gives us arclength. For functions $f(x)$, $g(y)$

   (a) Write $f(x)$ and $g(y)$ as parametric equations.

   (b) Use these parametric equations and the equation for speed to arrive at the formulas for (1) the arc length of $f(x)$ from $x = a$ to $x = b$, and (2) the arc length of $g(y)$ from $y = c$ to $y = d$.

Lecture Notes:

- Given $\begin{cases} x(t) \\ y(t) \end{cases}$, we can find $x'(t)$ and $y'(t)$. The slope of the curve at t is given by $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$.

- We can also consider how quickly the coordinate $\big(x(t), y(t)\big)$ is changing with respect to t. We call $\sqrt{(x'(t))^2 + (y'(t))^2}$ the **speed** at t. This looks similar to the distance equation for good reason—we want to know what distance the point is moving as t changes. (Aside: Since speed tells us how quickly the distance is changing, integrating speed will give us the total distance—otherwise known as arclength.)

- To find the tangent line to a curve, we compute $\dfrac{dy}{dx}$ as usual, but we have the additional step of moving back and forth between $x - y$ coordinates and our parameter t.

  1. Find the tangent line to $\begin{cases} x(t) = \sin(2t) \\ y(t) = \sin(t) + 1 \end{cases}$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ at $(0, 1)$.

     To compute $\dfrac{dy}{dx}$, we need to know what value of t corresponds to $(0, 1)$. Since $\sin(2t) = 1$ for $t = 0, \pi/2, -\pi/2$, in this range of t and $\sin(t) + 1 = 1$ only for $t = 0$, $(1, 0)$ is on the curve and corresponds to $t = 0$. We have $x'(t) = 2\cos(2t)$, $y'(t) = \cos(t)$. So, $\dfrac{dy}{dx} = \dfrac{\cos(t)}{2\cos(2t)}$. So at $(0, 1)$, the slope is $\dfrac{\cos(0)}{2\cos(2(0))} = \dfrac{1}{2}$. Then, we assemble the tangent line as usual: $(y - 1) = \frac{1}{2}(x - 0)$

  2. Find the tangent line to $\begin{cases} x(t) = 4t^2 \\ y(t) = e^t \end{cases}$, $-2 \leq t \leq 2$ at $t = 1$.

     We have $\dfrac{dy}{dx} = \dfrac{e^t}{8t}$. At $t = 1$, we have $\dfrac{e}{8}$. We want to assemble the tangent line, but need the x and y coordinates for that. $x(1) = 4$, $y(1) = e$. So our tangent line is: $y - e = e(x - 4)$

- Important Things to Look Out for:

  - If $y'(t) = 0$, the curve has a horizontal tangent line at t. If $x'(t) = 0$, the curve has a vertical tangent line at t.
  - If $\big(x(t_0), y(t_0)\big) = \big(x(t_1), y(t_1)\big)$ (i.e., the curve intersects itself), the curve doesn't necessarily have the same tangent line at t_0 as at t_1.
    Example (draw 2 tangent lines at origin):